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A class of proximity-sensitive measures of relative deprivation

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Abstract

We introduce a new class of generalized measures of relative deprivation. The class takes the form of a power mean of order p. A characteristic of the class is that depending on the value of the proximity-sensitive parameter p, the class is capable of accommodating both a decreasing weight (the case of $p > 1$), and an increasing weight (the case of $p \in (0,1)$) accorded to given changes in the incomes of the individuals who are wealthier than the reference individual, depending on their proximity in the income distribution to the reference individual.

Keywords: Income distribution; Relative deprivation; Sensitivity to income transfers between wealthier individuals; Sensitivity to the proximity of changes in others' incomes

JEL classification: D31; D33; D63; H23

1. Introduction

It is widely recognized that individuals feel stressed when their income (wealth) is lower than the income (wealth) of others with whom they naturally compare themselves (these "others" constitute the individuals' comparison group). The "relative deprivation" sensed by an individual can be measured in a variety of ways. The (income related) index that has become center stage is the aggregate of the excesses of the incomes of the other individuals in an individual's comparison group divided by the number of individuals in the individual's comparison group (essentially an operationalization of Runciman's 1966 relative deprivation concept by Yitzhaki, 1979; Hey and Lambert, 1980; Chakravarty, 1999; Ebert and Moyes, 2000; Bossert and D'Ambrosio, 2006; Stark and Hyll, 2011). An assumption made in both theoretical and empirical writings that have incorporated relative deprivation is that comparisons with others who are positioned to the right of the individual in the income distribution count equally: the income excesses of those who are close by and the income excesses of those who are farther away are accorded equal importance. However recent evidence (Obloj and Zenger, 2015; Quintana-Domeque and Wohlfart, 2016) indicates that people attach different importance to changes in incomes of individuals who are farther away in the income distribution than to changes in incomes of adjacent individuals.

In this paper we question the equal weights convention. We propose a general and flexible weighting protocol, based on the notion that the same importance need not be attached to changes in income of individuals who are placed at different distances from the individual whose relative deprivation is measured. Operationalizing the income shortfall approach via a set of axioms enables us to obtain a class of measures that has the form of a power mean of the excesses of the incomes of others, parameterized by a positive number p.

Several other generalizations of the index of relative deprivation have already been proposed: Chakravarty and Chakroborty (1984), Paul (1991), Wang and Tsui (2000), Bossert and D'Ambrosio (2007, 2014), and Esposito (2010). The main difference between five of these six contributions and the generalization presented in this paper is that the indices proposed by Chakravarty and Chakroborty (1984), Paul (1991), and Wang and Tsui (2000) are not derived from axioms; the perspective pursued by Esposito (2010) is not based on the income shortfall; and the index proposed by Bossert and D'Ambrosio (2007) adheres to the equal weights convention. Only the generalization offered by Bossert and D'Ambrosio (2014) derives axiomatically a class of proximity-sensitive measures of relative deprivation based on income shortfalls. Our approach follows in the steps of Bossert and D'Ambrosio (2014), yet it takes the analysis a step further. Whereas the Bossert and D'Ambrosio's (2014) index allows for only one type of proximity-sensitivity, our proposed RD_p class of measures is proximity-sensitive in a more general sense: right-hand side changes in income weigh differentially, depending on how distant they are in the income distribution, and this variation is exhibited by the value of the proximity-sensitive parameter *p*: for $p \in (0,1)$, the greater the distance, the smaller the impact of a given change in income on the relative deprivation sensed by the individual; for $p > 1$, the opposite effect applies.

As already noted, there can very well be situations in which people might be more disturbed by a given increase in income of an already relatively rich individual in their comparison group than by an equal increase in income of a not so rich individual in their comparison group. Thus, we derive a class of measures which, depending on the parameter *p*, can be applied to both types of sensitivity to the proximity of the incomes of others. Needless to say, the derived class of measures allows more nuanced analyses of settings in which relative deprivation considerations play a role. And, after all, if people need to be compensated for experiencing increased relative deprivation, the manner of calculating the index also matters greatly in the context of welfare-related policy formation.

In Section 2 we introduce a preference relation in the set of possible comparison groups, and we equip this relation with properties (axioms) that we consider natural for an ordering. We show that the only measure that fulfills the listed axioms is the index RD_p. In Section 3 we deal in some detail with the subset of the axioms that are related to the proximity-sensitivity property of RD_p. Section 4 concludes.

2. Axiomatization of order $p > 0$ of the relative deprivation sensed by an individual

We consider a population of $n+1$ individuals, where n is a positive integer. The income distribution of this population is $(z, \mathbf{x}) \in \mathbf{R}_+^{n+1}$, where z is the (non-negative) income of individual ω, and $\mathbf{x} = \left(x_1, \ldots, x_n\right)$ is the vector of (non-negative) incomes of the comparison group of ω. We denote $I_{\mathbf{x}} = \{i : x_i > z\}$, namely $I_{\mathbf{x}}$ is the subset of the comparison group $\mathbf{x}$ that consists of individuals whose incomes are higher than the income of ω. And we denote by Ω^{n+1} the set of vectors of (non-negative) incomes of individual ω and of the members of his comparison group: $(z, \mathbf{x}) \in \Omega^{n+1}$.

We introduce a binary relation $\pm$ on the set Ω^{n+1}. This relation will reflect an individual's preference for the level of relative deprivation arising from a comparison of his income z with the incomes of members of two different comparison groups: an individual will prefer a comparison group that makes him less relatively deprived. We denote by $\sim$ the symmetric part of $\pm$, and by $\succ$ the asymmetric part of $\pm$.

We begin with a set of axioms that are needed to ensure that comparisons with the incomes of other individuals are represented by non-negative income differences.

Focus axiom (Axiom F). Let $(z,\mathbf{x}),(z,\mathbf{y}) \in \Omega^{n+1}$ be such that $I_{\mathbf{x}} = I_{\mathbf{y}}$ and $x_i = y_i$ for every $i \in I_{\mathbf{x}}$. Then $(z,\mathbf{x}) \sim (z,\mathbf{y})$.

The Focus axiom requires the individual to be indifferent to the incomes of those who are poorer than him. The axiom reflects the fact that individual ω experiences relative deprivation only when he compares his income with incomes that are higher than his.

Translation Invariance axiom (Axiom TI). If $(z,\mathbf{x}) \in \Omega^{n+1}$ and $\delta \in [-\min\{z, x_1,..., x_n\}, \infty)$, then

$$\left(z, x_1, \dots, x_n\right) \sim \left(z+\delta, x_1+\delta, \dots, x_n+\delta\right).$$

Translation Invariance requires the index of relative deprivation to be indifferent to a positive transformation, applied to all incomes, provided that all incomes stay non-negative. Therefore, the axiom imposes a sensitivity of the relative deprivation measure not to the absolute income of an individual, but to the income differences between the incomes of others and his own income.

Monotonicity axiom (Axiom M). Let $\mathbf{x} = (x_1, \dots, x_i, \dots, x_n)$ and $\mathbf{y} = (x_1, \dots, x_i + \eta, \dots, x_n)$ for some $i \in \{1, \dots, n\}$ and $\eta > 0$. Then, if $x_i + \eta > z$, we have that $(z,\mathbf{x}) \succ (z,\mathbf{y})$.

The Monotonicity axiom requires an individual to be strictly more relatively deprived if a wealthier individual (meaning an individual whose income is higher) in his comparison group is made richer, and equally relatively deprived if a poorer individual is made richer yet remains (weakly) poorer. In addition by Axiom M, the larger the increase of the income of the wealthier individual, the larger the added relative deprivation experienced by individual ω.

Continuous Ordering axiom (Axiom CO). The relation $\succeq$ is a continuous linear ordering on Ω^{n+1} that can be represented by a continuous function (in the Euclidean metric on $\mathbf{R}^{n+1}$) $F : \Omega^{n+1} \mapsto [0, \infty)$ well-defined for all vectors $(z,\mathbf{x}) \in \Omega^{n+1}$, that is,

$$(z,\mathbf{x}) \pm (z,\mathbf{y}) \Leftrightarrow F(z,\mathbf{x}) \le F(z,\mathbf{y}).$$

Axiom CO requires the binary relation to be a continuous linear ordering that is represented by a continuous function that, in turn, is well-defined for all possible income distributions. To ensure focus on essentials, in the remainder of this paper we draw on this representation, thereby bypassing the need to recall Axiom CO explicitly.

Reflexivity axiom (Axiom R). If all the components of the vector $\mathbf{x}$ are equal, that is, if $\mathbf{x}=(x,\dots,x)$, then $F(z,\mathbf{x})=\max\{x-z,0\}$.

The Reflexivity Axiom requires that if individual ω compares his income with the incomes of the members of an "egalitarian" comparison group, then his relative deprivation with respect to this group is equal to the group's common income minus his own income, with a floor of zero.

Anonymity axiom (Axiom A). If $\mathbf{y}$ is a vector of incomes obtained from vector $\mathbf{x}$ by permutation of its components, then $(z,\mathbf{x})\sim(z,\mathbf{y})$.

The Anonymity axiom requires the binary relation to be indifferent to a permutation of the components of the reference vector. Thus, the axiom postulates an irrelevance of individual identities for the value of the index of relative deprivation.

Population Substitution Principle axiom (Axiom PSP). If $\mathbf{x}=\left(x_1,\dots,x_n\right)$ and $(z,\mathbf{x})\in\Omega^{n+1}$, then $\left(z,x_1,\dots,x_n\right)\sim\left(z,F\left(z,x_1\dots,x_k\right)+z,\dots,F\left(z,x_1\dots,x_k\right)+z,x_{k+1},\dots,x_n\right)$ for every $k\le n$.

In Axiom PSP we consider a subpopulation of the comparison group $\mathbf{x}$ consisting of k individuals (by Axiom A we have that this sub-population can be chosen arbitrarily). $F\left(z,x_1\dots,x_k\right)+z$ denotes their equivalent income, namely if income of every member of the sub-population is replaced by $F\left(z,x_1\dots,x_k\right)+z$ then the new vector of incomes and the vector $\mathbf{x}$ are equivalent.

Scale Invariance axiom (Axiom SI). Let $(z,\mathbf{x}),(z,\mathbf{y})\in\Omega^{n+1}$ and $\lambda>0$. If $(z,\mathbf{x})\sim(z,\mathbf{y})$, then $(\lambda z,\lambda\mathbf{x})\sim(\lambda z,\lambda\mathbf{y})$.

The Scale Invariance axiom requires the binary relation to be invariant to a rescaling of the incomes.

With the preceding axioms in place, we are ready to present our main result.

Theorem 1. If relation $\succeq$ on Ω^{n+1} satisfies axioms F, TI, CO, M, A, R, PSP, and SI, then there exists $p>0$ such that

$$F\left(z,\mathbf{x}\right)=\left(\frac{1}{n}\sum_{i=1}^{n}\left(\max\left\{x_i-z,0\right\}\right)^p\right)^{\frac{1}{p}}. \tag{1}$$

Proof. The proof is in the Appendix.

The result stated in Theorem 1 is not too surprising when we consider related work in social choice theory (Blackorby and Donaldson, 1982; Ebert, 1988). However, whereas the orderings in that related work are based on a macroeconomic approach (the perspective of the social planner), the ordering in Theorem 1 is with respect to the selected individual.

Subsequently, we denote the function in (1) by $RD_p(z,\mathbf{x})$ or, in short, by RD_p:

$$RD_p(z,\mathbf{x})\equiv\left(\frac{1}{n}\sum_{i=1}^{n}(\max\{x_i-z,0\})^p\right)^{\frac{1}{p}}=\left(\frac{1}{n}\sum_{i\in I_{\mathbf{x}}}(x_i-z)^p\right)^{\frac{1}{p}},$$

and we refer to this function as a (generalized) index of relative deprivation of order $p>0$.

3. The proximity-sensitivity of RD_p

In this section we introduce additional axioms that allow us to differentiate between the cases $p\in(0,1)$ and $p>1$, and we further indicate in what ways the RD_p index differs from the received index that assigns equal weights to all the income excesses. As already noted, for any *p* other than 1, the RD_p class of measures is proximity-sensitive, depending on the range of *p*.

We begin by referring to the sensitivity of RD_p to income changes of individuals who are to the right of ω in the income distribution, depending on their proximity to ω.

Rising Proximity-Sensitivity axiom (Axiom RPS). Let $\mathbf{x}=\left(x_1,\ldots,x_k,\ldots,x_l,\ldots,x_n\right)$, where $z\le x_k<x_l$, and $\mathbf{y}^1=(x_1,\ldots,x_k+\delta,\ldots,x_l,\ldots,x_n)$, $\mathbf{y}^2=(x_1,\ldots,x_k,\ldots,x_l+\delta,\ldots,x_n)$ where $\delta>0$. Then $\left(z,\mathbf{y}^2\right)\succ\left(z,\mathbf{y}^1\right)$.

Declining Proximity-Sensitivity axiom (Axiom DPS). Let $\mathbf{x}=\left(x_1,\ldots,x_k,\ldots,x_l,\ldots,x_n\right)$, where $z\le x_k<x_l$, and $\mathbf{y}^1=(x_1,\ldots,x_k+\delta,\ldots,x_l,\ldots,x_n)$, $\mathbf{y}^2=(x_1,\ldots,x_k,\ldots,x_l+\delta,\ldots,x_n)$ where $\delta>0$. Then $\left(z,\mathbf{y}^1\right)\succ\left(z,\mathbf{y}^2\right)$.

The Rising Proximity-Sensitivity axiom states that individual ω will sense more relative deprivation as a result of an increase in income of an individual who (placed to the right of ω in the income distribution) is closer to him than as a result of an equal increase in income of an individual who (placed to the right of ω in the income distribution) is farther away. The Declining Proximity-Sensitivity axiom states the opposite: the shorter the distance from z, the smaller the impact of the described income change on the relative deprivation sensed by ω.[1]

Theorem 2. If the relation $\pm$ on Ω^{n+1} satisfies the assumptions of Theorem 1, then:

(1) The relation $\pm$ satisfies Axiom DPS if and only if $F(z,\mathbf{x}) = RD_p(z,\mathbf{x})$ for $p > 1$;

(2) The relation $\pm$ satisfies Axiom RPS if and only if $F(z,\mathbf{x}) = RD_p(z,\mathbf{x})$ for $p \in (0,1)$.

Proof. The proof is in the Appendix.

Finally, we refer also to the sensitivity of RD_p to the transfer of income *between* individuals who are wealthier than ω. We present properties of the generalized index of relative deprivation to the effect that following the transfer, the positions (ordering) of the transferer and the transferee with respect to ω and with respect to each other do not change.

Progressive Transfer property (Property PT). Let $\mathbf{x} = (x_1,\ldots,x_n)$, where $z \le x_k < x_l$, $k,l \in \{1,\ldots,n\}$ and $0 < \delta \le \frac{x_l - x_k}{2}$. If $\mathbf{y} = (x_1,\ldots,x_k + \delta,\ldots,x_l - \delta,\ldots,x_n)$, then $(z,\mathbf{y}) \succ (z,\mathbf{x})$.

Regressive Transfer property (Property RT). Let $\mathbf{x} = (x_1,\ldots,x_n)$, where $z \le x_k < x_l$, $k,l \in \{1,\ldots,n\}$ and $0 < \delta \le x_k - z$. If $\mathbf{y} = (x_1,\ldots,x_k - \delta,\ldots,x_l + \delta,\ldots,x_n)$, then $(z,\mathbf{y}) \succ (z,\mathbf{x})$.

The Progressive Transfer property implies that a top-down transfer is welcomed by ω: a population (a distribution of incomes) after such a transfer is preferred by individual ω to a population prior to the transfer. The Regressive Transfer property implies the opposite:

[1] A justification for the Declining Proximity-Sensitivity axiom is that individual ω may be tolerant of an income gain by someone on a similar income rung, but not so when someone already significantly richer than himself becomes even richer. This tolerance/displeasure dichotomy could arise from a basic notion of fairness: when looking to the right, ω considers relatively poor "neighbors" more deserving of an income rise than relatively rich "neighbors." The viability of such reasoning is not in contradiction with the stance taken in received studies. The index proposed by Esposito (2010) incorporates the consideration of an upper boundedness of a relative deprivation measure, which can be perceived as a boundary placed on the space that accommodates the reference group. The empirical study by Quintana-Domeque and Wohlfart (2016) supports the prevalence of such a boundary.

individual ω prefers a population (a distribution of incomes) after a bottom-up transfer to a population prior to the transfer.

Corollary 1. If the relation $\pm$ on Ω^{n+1} satisfies the assumptions of Theorem 1, then:

(1) Property PT is equivalent to Axiom DPS;

(2) Property RT is equivalent to Axiom RPS.

Proof. The proof is in the Appendix.

Theorem 2 and Corollary 1 reveal that the RD_p index exhibits two types of proximity-sensitivity. The Declining Proximity-Sensitivity axiom and the Rising Proximity-Sensitivity axiom refer to sensitivity to the proximity of changes of incomes of wealthier individuals, whereas the Progressive Transfer property and the Regressive Transfer property refer to sensitivity to income transfers between wealthier individuals.

4. Conclusion

We introduced a new class of measures of relative deprivation, RD_p, based on a preference relation defined on the set of vectors of incomes. RD_p is a generalization of the standard index of relative deprivation in that for any positive value of the proximity-sensitive parameter *p* different from one, the class exhibits sensitivity to the proximity of changes in the incomes of individuals whose incomes are higher than the income of the reference individual. The class is capable of accommodating the case of decreasing weights and the case of increasing weights accorded to given changes in incomes that are higher than the income of the reference individual (the individual whose relative deprivation is measured). Theoretically, a rationale can be provided in support of each of these cases. It will therefore be of considerable interest to identify empirically settings in which the impact on relative deprivation is represented by values of *p* that are smaller than one and settings in which the impact on relative deprivation is represented by values of *p* that are greater than one.

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Appendix: Proofs of Theorem 1, Theorem 2, and Corollary 1

To prove Theorem 1, we first make three remarks, and we present and prove two lemmas.

Remark 1. To simplify notation, let the distribution of incomes in population $(z, \mathbf{x})$ be such that $x_1 \le x_2 \le \ldots \le x_i \le z \le x_{i+1} \le \ldots \le x_n$. From the Focus axiom we know that $(z, x_1, \ldots, x_n) \sim (z, z, \ldots, z, x_{i+1}, \ldots, x_n)$. By invoking the Translation Invariance axiom with $\delta = -z$, we see that $(z, z, \ldots, z, x_{i+1}, \ldots, x_n) \sim (0, 0, \ldots, 0, x_{i+1} - z, \ldots, x_n - z)$. From these two facts we see that

$$(z, x_1, \ldots, x_n) \sim (0, 0, \ldots, 0, x_{i+1} - z, \ldots, x_n - z).$$

Remark 1 states that individual ω compares his income z with the incomes of members of his comparison group $\mathbf{x}$ and experiences relative deprivation arising from differences between the incomes of members of his comparison group $\mathbf{x}$ whose incomes are higher than his and his own income. Thus, we define a function $\Delta : \Omega^{n+1} \mapsto \Omega_z^n$, where

$$\Delta(z, \mathbf{x}) \equiv \left(\max\{x_1 - z, 0\}, \ldots, \max\{x_n - z, 0\}\right).$$

The function $\Delta(\cdot)$ maps the set of vectors of non-negative incomes Ω^{n+1} onto the set of vectors of non-negative differences between those incomes and income z, denoted by $\Omega_z^n \subset \mathbf{R}^n$.

Remark 2. From axioms F, TI and CO (and Remark 1) we get that there is a continuous function $G : \Omega_z^n \mapsto [0, \infty)$ such that $F(z, \mathbf{x}) = G(\Delta(z, \mathbf{x}))$.

Remark 2 states that function $F(\cdot)$ is formed by a composition of function $\Delta(\cdot)$, which reflects the fact that individual ω compares his income z with the incomes of members of his comparison group $\mathbf{x}$, and of function $G(\cdot)$.

Lemma 1. If the relation $\succeq$ on Ω^{n+1} satisfies axioms F, TI, CO, M, A, R, and PSP, then there exists a continuous, increasing function $g:[0,\infty)\mapsto[0,\infty)$ such that for every $y\in\Omega_z^n$

$$G(y)=g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}g(y)\right).$$

Proof. Lemma 1 is a well-known property due to Kolmogorov (1930), as proved by Tikhomirov (1991, p. 144) which we, thus, state without providing proof.

From Lemma 1 we know that if axioms F, TI, CO, M, A, R, and PSP are fulfilled, then function $G(\cdot)$ has the form of a quasi-arithmetic mean.

Remark 3. From Remark 2 and Lemma 1 we see that for every $(z,\mathbf{x})\in\Omega^{n+1}$

$$F(z,\mathbf{x})=g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}g\left(\max\{x_i-z,0\}\right)\right). \tag{A1}$$

The function in (A1) constitutes a general form of a class of measures of relative deprivation satisfying the axioms listed in Lemma 1.

Lemma 2. (Homogeneity of degree 1)**.** If the relation $\succeq$ on Ω^{n+1} satisfies axioms F, TI, CO, R, and SI, then for every $\lambda>0$, $F\left(\lambda\cdot(z,\mathbf{x})\right)=\lambda\cdot F(z,\mathbf{x})$.

Proof of Lemma 2. Let $(z,\mathbf{x})\in\Omega^{n+1}$, and let $\mathbf{y}=\left(F(z,\mathbf{x})+z,\dots,F(z,\mathbf{x})+z\right)$. Because all the components of the vector $\mathbf{y}$ are identical, Axiom R implies that $F(z,\mathbf{y})=\max\left\{F(z,\mathbf{x}),0\right\}=F(z,\mathbf{x})$. Therefore, we get that vectors $(z,\mathbf{x})$ and $(z,\mathbf{y})$ are equivalent. By Axiom SI we see that vectors $(\lambda z,\lambda\mathbf{x})$ and $(\lambda z,\lambda\mathbf{y})$ are also equivalent for every $\lambda>0$. Therefore, from axioms CO and R we get that

$$\begin{aligned}F\left(\lambda\cdot(z,\mathbf{x})\right)&=F\left(\lambda z,\lambda\mathbf{x}\right)=F\left(\lambda z,\lambda\mathbf{y}\right)\\&=F\left(\lambda z,\lambda\cdot F(z,\mathbf{x})+\lambda z,\dots,\lambda\cdot F(z,\mathbf{x})+\lambda z\right)=\lambda\cdot F(z,\mathbf{x}).\end{aligned}$$

Q.E.D.

For ease of reference, we replicate the theorems and the corollary.

Theorem 1. If relation ± on Ω^{n+1} satisfies axioms F, TI, CO, M, A, R, PSP, and SI, then there exists $p>0$ such that

$$F(z,\mathbf{x})=\left(\frac{1}{n}\sum_{i=1}^{n}\left(\max\left\{x_i-z,0\right\}\right)^p\right)^{\frac{1}{p}}.$$

Proof of Theorem 1. From (A1) we know that

$$F\left(\lambda\cdot(z,\mathbf{x})\right)=g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}g\left(\lambda\cdot\max\left\{x_i-z,0\right\}\right)\right).$$

From Lemma 2 it follows that

$$g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}g\left(\lambda\cdot\max\left\{x_i-z,0\right\}\right)\right)=\lambda\cdot g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}g\left(\max\left\{x_i-z,0\right\}\right)\right)$$

or, equivalently, that

$$\frac{1}{n}\sum_{i=1}^{n}g\left(\lambda\cdot\max\left\{x_i-z,0\right\}\right)=g\left(\lambda\cdot g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}g\left(\max\left\{x_i-z,0\right\}\right)\right)\right). \tag{A2}$$

We implicitly define $t_i\geq 0$ for $i=1,\ldots,n$ by $g^{-1}\left(t_i\right)=\max\left\{x_i-z,0\right\}$, and from (A2) we see that

$$\frac{1}{n}\sum_{i=1}^{n}g\left(\lambda\cdot g^{-1}(t_i)\right)=g\left(\lambda\cdot g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}t_i\right)\right). \tag{A3}$$

We abbreviate $h_\lambda(t)\equiv g\left(\lambda\cdot g^{-1}\left(t\right)\right)$, and rewrite (A3) in an equivalent form:

$$\frac{1}{n}\sum_{i=1}^{n}h_\lambda\left(t_i\right)=h_\lambda\left(\frac{1}{n}\sum_{i=1}^{n}t_i\right). \tag{A4}$$

The solution of the functional equation (A4) is given by:

$$h_\lambda(t)=c(\lambda)\cdot t+a(\lambda)$$

(as per Theorem 2 in Aczél, 1966, p. 48), where $c,a:[0,\infty)\mapsto\mathbf{R}$. Therefore,

$$\begin{aligned}h_\lambda\left(t_i\right)&=g\left(\lambda\cdot g^{-1}\left(t_i\right)\right)=g\left(\lambda\cdot\max\left\{x_i-z,0\right\}\right)\\&=c(\lambda)\cdot t_i+a(\lambda)=c(\lambda)\cdot g\left(\max\left\{x_i-z,0\right\}\right)+a(\lambda).\end{aligned}$$

The functional equation

$$g\left(\lambda\cdot\max\left\{x_i-z,0\right\}\right)=c(\lambda)\cdot g\left(\max\left\{x_i-z,0\right\}\right)+a(\lambda)$$

has a solution given by (as per Theorem 2.7.3 in Eichhorn, 1978)

$$g(x)=\beta\cdot x^p+\delta,\qquad c(\lambda)=\lambda^p,\qquad a(\lambda)=\delta(1-\lambda^p), \tag{A5}$$

or by

$$g(x)=\beta\cdot\ln(x)+\delta,\qquad c(\lambda)\equiv 1,\qquad a(\lambda)=\beta\cdot\ln(\lambda), \tag{A6}$$

where $p,\beta\neq 0$, $\delta\in\mathbf{R}$ are arbitrary constants. Because $g^{-1}\left(t_i\right)=\max\left\{x_i-z,0\right\}=0$ for $x_i\leq z$, the function g in (A5) for $p<0$, and the function g in (A6) are not well defined for $x_i\leq z$. Thus, we see that

$$g(x)=\beta\cdot x^p+\delta,\ c(\lambda)=\lambda^p,\ \text{and}\ a(\lambda)=\delta(1-\lambda^p),\ \text{for}\ p>0. \tag{A7}$$

By inserting (A7) in (A1) we see that

$$F(z,\mathbf{x})=g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}g\left(\max\left\{x_i-z,0\right\}\right)\right)=\left(\frac{\frac{1}{n}\cdot\sum_{i=1}^{n}\Big(\beta\cdot\left(\max\left\{x_i-z,0\right\}\right)^p+\delta\Big)-\delta}{\beta}\right)^{\frac{1}{p}}$$

$$=\left(\frac{1}{n}\sum_{i=1}^{n}\left(\max\left\{x_i-z,0\right\}\right)^p\right)^{\frac{1}{p}},$$

for $p>0$.

In sum, the only possible representation of the relative deprivation sensed by an individual satisfying axioms F, TI, CO, M, A, R, PSP and SI is given by

$$F(z,\mathbf{x})=\left(\frac{1}{n}\sum_{i=1}^{n}\left(\max\left\{x_i-z,0\right\}\right)^p\right)^{\frac{1}{p}},$$

for $p>0$. Q.E.D.

Theorem 2. If the relation $\pm$ on Ω^{n+1} satisfies the assumptions of Theorem 1, then:

(1) The relation $\pm$ satisfies Axiom DPS if and only if $F(z,\mathbf{x})=RD_p(z,\mathbf{x})$ for $p>1$;

(2) The relation $\pm$ satisfies Axiom RPS if and only if $F(z,\mathbf{x})=RD_p(z,\mathbf{x})$ for $p\in(0,1)$.

Proof of Theorem 2. Let $\mathbf{x}=\left(x_1,\ldots,x_k,\ldots,x_l,\ldots,x_n\right)$, where $z<x_k<x_l$, be the comparison group of individual ω, and let $\mathbf{y}^1=\left(x_1,\ldots,x_k+\delta,\ldots,x_l,\ldots,x_n\right)$ and $\mathbf{y}^2=\left(x_1,\ldots,x_k,\ldots,x_l+\delta,\ldots,x_n\right)$ be two comparison groups obtained from $\mathbf{x}$ by increasing by the same amount the incomes of individuals k and l, respectively. From the definition of RD_p we see that

$$RD_p\left(z,\mathbf{y}^1\right)=n^{-\frac{1}{p}}\left((x_k-z+\delta)^p+\sum_{i\neq k}(\max\{x_i-z,0\})^p\right)^{\frac{1}{p}}, \tag{A8}$$

and that

$$RD_p\left(z,\mathbf{y}^2\right)=n^{-\frac{1}{p}}\left((x_l-z+\delta)^p+\sum_{i\neq l}(\max\{x_i-z,0\})^p\right)^{\frac{1}{p}}. \tag{A9}$$

The relation ± on Ω^{n+1} satisfies Axiom DPS if $\left(z,\mathbf{y}^1\right)\succ\left(z,\mathbf{y}^2\right)$. From Axiom CO we get that $\left(z,\mathbf{y}^1\right)\succ\left(z,\mathbf{y}^2\right)$ if and only if $RD_p\left(z,\mathbf{y}^1\right)<RD_p\left(z,\mathbf{y}^2\right)$, and by (A8) and (A9) this last inequality can be presented in an equivalent form as

$$(x_l-z)^p+(x_k-z+\delta)^p<(x_l-z+\delta)^p+(x_k-z)^p,$$

or as

$$\frac{(x_k-z+\delta)^p-(x_k-z)^p}{\delta}<\frac{(x_l-z+\delta)^p-(x_l-z)^p}{\delta}. \tag{A10}$$

Because $x_k<x_l$ and because for $p>1$ the function $x\mapsto x^p$ is strictly convex, we infer that (A10) is fulfilled if and only if $p>1$. Thus, we have shown that part (1) of the theorem holds.

Similarly, part (2) of the theorem is a consequence of the strict concavity of the function $x\mapsto x^p$ for $p\in(0,1)$. Q.E.D.

Corollary 1. If the relation ± on Ω^{n+1} satisfies the assumptions of Theorem 1, then:

(1) Property PT is equivalent to Axiom DPS;

(2) Property RT is equivalent to Axiom RPS.

Proof of Corollary 1. We go first to part (1) of the corollary. Let $\mathbf{x}=\left(x_1,\ldots,x_k,\ldots,x_l,\ldots,x_n\right)$, where $z<x_k<x_l$, be the comparison group of individual ω, and let $\mathbf{y}=\left(x_1,\ldots,x_k+\delta,\ldots,x_l-\delta,\ldots,x_n\right)$, where $\delta<\dfrac{x_l-x_k}{2}$, be a comparison group obtained from $\mathbf{x}$ by a progressive transfer of the amount δ. The relative deprivation of order *p* sensed by individual w prior to the transfer is

$$RD_p(z,\mathbf{x})=\left(\frac{1}{n}\sum_{i=1}^{n}\left(\max\left\{x_i-z,0\right\}\right)^p\right)^{\frac{1}{p}}. \tag{A11}$$

The relative deprivation of order *p* sensed by individual ω following the transfer is

$$RD_p(z,\mathbf{y})=\left[\frac{1}{n}\left(\sum_{\substack{i=1\\ i\neq k,i\neq l}}^{n}(\max\{x_i-z,0\})^p+(x_k-z+\delta)^p+(x_l-z-\delta)^p\right)\right]^{\frac{1}{p}}. \tag{A12}$$

Because the relation $\succeq$ on Ω^{n+1} satisfies Property PT, we get that $(z,\mathbf{y})\succ(z,\mathbf{x})$. From Axiom CO we know that individual ω prefers constellation $(z,\mathbf{y})$ to constellation $(z,\mathbf{x})$ if and only if $RD_p(z,\mathbf{y})<RD_p(z,\mathbf{x})$. By comparing (A11) and (A12), we can present this last inequality in an equivalent form as

$$(x_k - z + \delta)^p + (x_l - z - \delta)^p < (x_k - z)^p + (x_l - z)^p,$$

or as

$$\frac{(x_k - z + \delta)^p - (x_k - z)^p}{\delta} < \frac{(x_l - z)^p - (x_l - z - \delta)^p}{\delta}. \tag{A13}$$

Because $x_k - z + \delta < x_l - z - \delta$, we see that inequality (A13) is fulfilled if and only if the function $x \mapsto x^p$ is strictly convex, that is, if and only if $p > 1$. Thus, we see that Property PT holds if and only if $F(\mathbf{x}_z) = RD_p(z, \mathbf{x})$ for $p > 1$. From part (1) of Theorem 2 we see that Property PT is equivalent to Axiom DPS.

Similarly, part (2) of the corollary is a consequence of the strict concavity of function $x \mapsto x^p$ for $p \in (0,1)$ and of part (2) of Theorem 2. Q.E.D.

References

Aczél, János (1966). *Lectures on functional equations and their applications*. New York: Academic Press.

Blackorby, Charles and Donaldson, David (1982). "Ratio-scale and translation-scale full interpersonal comparability without domain restrictions: Admissible social-evaluation functions." *International Economic Review* 23(2): 249-268.

Bossert, Walter and D'Ambrosio, Conchita (2006). "Reference groups and relative deprivation." *Economics Letters* 90(3): 421-426.

Bossert, Walter and D'Ambrosio, Conchita (2007). "Dynamic measures of individual deprivation." *Social Choice Welfare* 28(1): 77-88.

Bossert, Walter and D'Ambrosio, Conchita (2014). "Proximity-sensitive individual deprivation measures." *Economics Letters* 122(2): 125-128.

Chakravarty, Satya R. (1999). "Measuring inequality: An axiomatic approach." In Silber J. (Ed.), *Handbook of Income Inequality Measurement*. Kluwer: Dordrecht, pp. 163-186.

Chakravarty, Satya R. and Chakraborty, A.B. (1984). "On indices of relative deprivation." *Economics Letters* 14(2): 283-287.

Ebert, Udo (1988). "Measurement of inequality: An attempt at unification and generalization." *Social Choice and Welfare*, 5(2/3): 147-169.

Ebert, Udo and Moyes, Patrick (2000). "An axiomatic characterization of Yitzhaki's index of individual deprivation." *Economics Letters* 68(3): 263-270.

Eichhorn, Wolfgang (1978). *Functional equations in economics*. Reading: Addison-Wesley.

Esposito, Lucio (2010). "Upper boundedness for the measurement of relative deprivation." *Review of Income and Wealth* 56(3): 632-639.

Hey, John D. and Lambert, Peter J. (1980). "Relative deprivation and the Gini coefficient: Comment." *Quarterly Journal of Economics* 95(3): 567-573.

Kolmogorov, Andrei N. (1930). "Sur la notion de la moyenne." *Atti della Accademia Nazionale dei Lincei*, Rendiconti 12(9): 388-391.

Obloj, Tomasz and Zenger, Todd (2015). "Incentives, social comparison costs, and the proximity of envy's object." HEC Paris Research Paper No. SPE-2015-1085.

Paul, Satya (1991). "An index of relative deprivation." *Economics Letters* 31(3): 337-341.

Quintana-Domeque, Climent and Wohlfart, Johannes (2016). "Relative concerns for consumptions at the top: An intertemporal analysis for the UK." *Journal of Economic Behavior and Organization* 129: 172-194.

Runciman, Walter G. (1966). *Relative Deprivation and Social Justice*. University of California Press: Berkeley.

Stark, Oded and Hyll, Walter (2011). "On the economic architecture of the workplace: Repercussions of social comparisons among heterogeneous workers." *Journal of Labor Economics* 29(2): 349-375.

Tikhomirov, Vladimir M. (1991). *Selected Works of A. N. Kolmogorov. Volume I: Mathematics and Mechanics*. Springer: Dordrecht.

Wang, You-qiang and Tsui, Kai-yuen (2000). "A new class of deprivation-based generalized Gini indices." *Economic Theory* 16(2): 363-377.

Yitzhaki, Shlomo (1979). "Relative deprivation and the Gini coefficient." *Quarterly Journal of Economics* 93(2): 321-324.